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EPFL, fall semester 2024
AG II - Schemes and sheaves

Exercises – week 2

Exercise 1. *Sheaves of abelian groups.* Let X be a topological space. Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphisms of sheaves of abelian groups.

(1) Let $\ker(\varphi)$ and $\mathrm{im}(\varphi)$ be respectively the kernel sheaf and the image sheaf.[1] Show that for every $x \in X$, one can define natural maps which are isomorphisms

$$\ker(\varphi)_x \to \ker(\varphi_x) \text{ and } \mathrm{im}(\varphi)_x \to \mathrm{im}(\varphi_x).$$

(2) Show that φ is an injective morphism of sheaves (resp. surjective morphism of sheaves) if and only if for every $x \in X$ the morphism of abelian groups $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective).
(3) Show that φ is a surjective morphism of sheaves if and only if for every U open and $s \in \mathcal{G}(U)$, there exists an open cover $U = \bigcup U_i$ and sections $t_i \in \mathcal{F}(U_i)$ with $\varphi(t_i) = s_{U_i}$.
(4) Show that the natural map $\mathrm{im}(\varphi) \to \mathcal{G}$ is injective.
(5) Show that φ is an isomorphism if and only if it is an injective morphism of sheaves and a surjective morphism of sheaves.
(6) Let $f = X \to *$ be the unique morphism to the point. Show that $f_* = \Gamma(X, -) : \mathrm{Sh}_{\mathrm{Ab}}(X) \to \mathrm{Ab}$ is left-exact. Give an example to show that f_* is not right-exact in general.

Exercise 2. *Gluing sheaves.* Let X be a topological space and $\bigcup U_i = X$ an open cover of X. Let $(\mathcal{F}_i \in \mathrm{Sh}(U_i), \varphi_{ij})$ be a collection of sheaves on $\mathrm{Sh}(U_i)$ together with isomorphisms

$$\varphi_{ij} : \mathcal{F}_{i|U_{ij}} \xrightarrow{\sim} \mathcal{F}_{j|U_{ij}}$$

in $\mathrm{Sh}(U_{ij})$ satisfying for each i that $\mathrm{id} = \varphi_{ii}$ and for each i, j, k the following *cocycle condition* $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$.

Show that there exists a unique[2] sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ with maps $\psi_i : \mathcal{F}_{|U_i} \to \mathcal{F}_i$ with the following universal property: for all sheaves $\mathcal{G} \in \mathrm{Sh}(X)$ we have a a bijection

$$\mathrm{Hom}(\mathcal{G}, \mathcal{F}) \cong \left\{ (\mathcal{G}_{|U_i} \xrightarrow{f_i} \mathcal{F}_i) \in \prod_i \mathrm{Hom}(\mathcal{G}_{|U_i}, \mathcal{F}_i) \mid \text{s.t. for all } i, j : \varphi_{ij} f_i = f_j \right\}$$

given by $f \mapsto \psi_i f_{|U_i}$.

Show furthermore that ψ_i are isomorphisms.

[1] The kernel sheaf is the kernel presheaf but the image sheaf is the *sheafification* of the image presheaf.

[2] up to isomorphism.

Remark. Can you see how the last exercise resembles the following statement: "$U \mapsto \mathrm{Sh}(U)$ is a sheaf"?

Exercise 3. *Inverse image.* Let $f\colon X \to Y$. Let $\mathcal{F} \in \mathrm{Sh}(Y)$. We define the presheaf on X

$$f^\sharp \mathcal{F}(U) = \varinjlim_{V \supset f(U)} \mathcal{F}(V).$$

(1) Show that if $f\colon * \to X$ is a point $x \in X$ then $f^\sharp \mathcal{F} = \mathcal{F}_x$.
(2) Show that if $y = f(x)$ then there is a natural isomorphism

$$(f^\sharp \mathcal{F})_x \to \mathcal{F}_y.$$

(3) Show that if f is an open immersion, then $f^\sharp$ is a sheaf.
(4) Find an example of map of topological spaces $f : X \to Y$ and a sheaf $\mathcal{F}$ on Y such that $f^\sharp \mathcal{F}$ is *not* a sheaf.
(5) Let $f^{-1}\mathcal{F}$ be the sheafification of $f^\sharp \mathcal{F}$. We call this sheaf the *inverse image* of $\mathcal{F}$. Show that the $f^{-1} \dashv f_*$[3] meaning that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{F}, f_*\mathcal{G}).$$

Exercise 4. *Localization* Let R be a ring. Let S be a multiplicative subset.

(1) Describe the points of $\mathrm{Spec}(S^{-1}R)$. If $\mathfrak{p} \in \mathrm{Spec}(R)$ show that $\mathrm{Spec}(R_\mathfrak{p})$ is the intersection of all opens containing $\mathfrak{p}$.
(2) Let M be an R-module and $I \supseteq R$ an ideal. Show that there is an isomorphism

$$S^{-1}(M/I) \cong (S^{-1}M)/(IS^{-1}M).$$

(3) Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and $I \leq R$ and ideal. When

$$(R/I)_\mathfrak{p} = 0 \quad ?$$

Can you interpret this geometrically?
(4) Let R be integral. Identify the image of the injective map $S^{-1}R \to \mathrm{Frac}(R)$.
(5) Let $R = \mathbb{Z}[x]$. Describe the localization at the maximal ideal (p, x).

Exercise 5. *Affine schemes are quasi-compact.* Let R be a ring. Show that $\mathrm{Spec}(R)$ is quasi-compact.[4] Deduce that the underlying topological space of any (affine) scheme has a basis of quasi-compact open subsets.

Exercise 6. *Connected affine schemes.* We say that a ring R is *connected* if for all $a, b \in R$ if

$$a + b = 1 \text{ and } ab = 0$$

then exactly one of the two elements is non-zero.

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[3] We say that f^{-1} is *left adjoint to* f_*

[4] A topological space X is *quasi-compact* if every open cover of X can be refined to a finite cover.

(1) Show that it is equivalent to the fact there is exactly two idempotents (namely 0 and 1) in the ring R.
(2) Show that R is connected if and only if $\mathrm{Spec}(R)$ is connected.

Exercise 7. *Stalks, morphisms and cotangent spaces*

(1) Let $X \to Y$ be a continuous map between topological spaces, and $\mathcal{F}$ a sheaf on X. Let $x \in X$ and $y = f(x)$. Show that there is a natural map
$$(f_*\mathcal{F})_y \to \mathcal{F}_x.$$
Remark. This is used to define the *induced map on local rings* of a map of locally ringed spaces. Namely if $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is one, with $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$, the induced map on local rings is for $x = f(y)$,
$$\mathcal{O}_{Y,y} \xrightarrow{f^\sharp_y} (f_*\mathcal{O}_X)_y \to \mathcal{O}_{X,x}.$$
(2) Let R be an integral domain. Consider $\varphi\colon R[x, y] \to R[x, y]$ defined by $x \mapsto xy$ and $y \mapsto y$. Consider
$$f\colon \mathrm{Spec}(R[x, y]) \to \mathrm{Spec}(R[x, y])$$
the induced map on associated affine schemes.[5] Show that for all $\lambda \in R$ we have $f((x - \lambda, y)) = (x, y)$.
(3) Let now $R = k$ a field. With point (1) and the remark there is induced map on local rings
$$k[x, y]_{(x,y)} \to k[x, y]_{(x-\lambda,y)}.$$
We write $\mathfrak{m}_{(0,0)} := \mathfrak{m}_{(x,y)}$ and $\mathfrak{m}_{(\lambda,0)} := \mathfrak{m}_{(x-\lambda,y)}$ for the maximal ideals of these local rings. Understand the induced k-linear map
$$\mathfrak{m}_{(0,0)}/\mathfrak{m}^2_{(0,0)} \to \mathfrak{m}_{(\lambda,0)}/\mathfrak{m}^2_{(\lambda,0)}.$$
This mean the following: find a k-basis of these vector spaces and describe the matrix of the map in term of your chosen basis.
Remark. We will later see that these vector spaces are the *cotangent spaces* at $(0, 0)$ and $(\lambda, 0)$ respectively and that the map that you studied is *the precompostion by the differential of f at these points.*

Exercise to hand in. *Induced map on* Spec (Due Sunday September 29, 18:00)
Please write your solution in TEX.
Let $f : R \to S$ be a ring homomorphism. Denote the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ by $f^\#$.

(1) Show that the closure of the image of $f^\#$ is $V(\ker(f))$. Find a ring theoretic property of f which is equivalent to the denseness of image of $f^\#$ in $\mathrm{Spec}(R)$.
(2) Find an example of a ring map where the image of the induced map on Spec is not closed. Find an example where the image is closed.

[5] Recall that the induced map on Spec is given by the *preimage* φ^{-1}

(3) Take $\mathfrak{q} \in \operatorname{Spec}(R)$. Set W to be the multiplicative subset $R \setminus \mathfrak{q}$. Prove that $(f^{\#})^{-1}(\{\mathfrak{q}\})$, as a set, is the underlying topological space of $\operatorname{Spec}(f(W)^{-1}(S/\mathfrak{q}S))$. Prove that

$$f(W)^{-1}(S/\mathfrak{q}S) \cong (R/\mathfrak{q})_{\mathfrak{q}} \otimes_R S$$

as rings. Here $\mathfrak{q}S$ denotes the ideal of S generated by $f(\mathfrak{q})$.

(4) Let $\iota : \mathbb{Z} \to \mathbb{Z}[X]$ be the inclusion map. Given a prime ideal $p\mathbb{Z}$ of $\mathbb{Z}$, describe $(\iota^{\#})^{-1}(p\mathbb{Z})$.

(5) Denote the algebraic closure of $\mathbb{Q}$ by $\overline{\mathbb{Q}}$. Let $\iota : \mathbb{Q}[X] \to \overline{\mathbb{Q}}[X]$ be the inclusion. Given an irreducible polynomial $g \in \mathbb{Q}[X]$, describe $(\iota^{\#})^{-1}(g\mathbb{Q}[X])$. Recall that for an irreducible polynomial g, the ideal $g\mathbb{Q}[X]$ is a prime ideal.

(6) Explain how Hilbert's Nullstellensatz gives a set-theoretic injection of $\mathbb{C}^2$ onto the closed points of $\operatorname{Spec}(\mathbb{C}[X,Y])$. Then, let $g \in \mathbb{C}[X,Y]$ be given. Show that there is a finite set of points $T \subseteq \operatorname{Spec}(\mathbb{C}[X,Y])$, such that the closed points in the closure $\overline{T}$ in $\operatorname{Spec}(\mathbb{C}[X,Y])$ are exactly the zeroes of g in $\mathbb{C}^2 \subseteq \operatorname{Spec}(\mathbb{C}[X,Y])$. Describe the smallest such T in terms of g.